Ill-conditioned Matrices

Consider systems

$$\begin{cases} x + y = 2 \\ x + 1.001y = 2 \end{cases} \quad \text{and} \quad \begin{cases} x + y = 2 \\ x + 1.001y = 2.001 \end{cases}$$

The system on the left has solution $x = 2, y = 0$ while the one on the right has solution $x = 1, y = 1$. The coefficient matrix is called *ill-conditioned* because a small change in the constant coefficients results in a large change in the solution. A *condition number*, defined in more advanced courses, is used to measure the degree of ill-conditioning of a matrix (≈ 4004 for the above).

In the presence of rounding errors, ill-conditioned systems are inherently difficult to handle. When solving systems where round-off errors occur, one must avoid ill-conditioned systems whenever possible; this means that the usual row reduction algorithm must be modified.

Consider the system:

$$\begin{cases} .001x + y = 1 \\ x + y = 2 \end{cases}$$

We see that the solution is $x = 1000/999 \approx 1, y = 998/999 \approx 1$ which does not change much if the coefficients are altered slightly (condition number ≈ 4).

The usual row reduction algorithm, however, gives an ill-conditioned system. Adding a multiple of the first to the second row gives the system on the left below, then dividing by -999 and rounding to 3 places on $998/999 = .99899 \approx 1.00$ gives the system on the right:

$$\begin{cases} .001x + y = 1 \\ -999y = -998 \end{cases} \qquad \begin{cases} .001x + y = 1 \\ y = 1.00 \end{cases}$$

The solution for the last system is $x = 0, y = 1$ which is wildly inaccurate (and the condition number is ≈ 2002).

This problem can be avoided using **partial pivoting**. Instead of pivoting on the first non-zero element, **pivot on the largest pivot (in absolute value) among those available in the column**.

In the example above, pivot on the x, which will require a permute first:

$$\begin{cases} x + y = 2 \\ .001x + y = 1 \end{cases} \qquad \begin{cases} x + y = 2 \\ .999y = .998 \end{cases} \qquad \begin{cases} x + y = 2 \\ y = 1.00 \end{cases}$$

where the third system is the one obtained after rounding. The solution is a fairly accurate $x = 1.00, y = 1.00$ (and the condition number is 4).