Ill-conditioned Matrices

Consider systems

\[
\begin{align*}
\begin{cases}
x + y &= 2 \\
x + 1.001y &= 2
\end{cases}
\quad \text{and} \quad
\begin{cases}
x + y &= 2 \\
x + 1.001y &= 2.001
\end{cases}
\end{align*}
\]

The system on the left has solution \(x = 2, y = 0\) while the one on the right has solution \(x = 1, y = 1\). The coefficient matrix is called \textit{ill-conditioned} because a small change in the constant coefficients results in a large change in the solution. A \textit{condition number}, defined in more advanced courses, is used to measure the degree of ill-conditioning of a matrix (\(\approx 4004\) for the above).

In the presence of rounding errors, ill-conditioned systems are inherently difficult to handle. When solving systems where round-off errors occur, one must avoid ill-conditioned systems whenever possible; this means that the usual row reduction algorithm must be modified.

Consider the system:

\[
\begin{align*}
\begin{cases}
.001x + y &= 1 \\
x + y &= 2
\end{cases}
\end{align*}
\]

We see that the solution is \(x = 1000/999 \approx 1, y = 998/999 \approx 1\) which does not change much if the coefficients are altered slightly (condition number \(\approx 4\)).

The usual row reduction algorithm, however, gives an ill-conditioned system. Adding a multiple of the first to the second row gives the system on the left below, then dividing by \(-999\) and rounding to 3 places on \(998/999 = .99899 \approx 1.00\) gives the system on the right:

\[
\begin{align*}
\begin{cases}
.001x + y &= 1 \\
-999y &= -998
\end{cases}
\quad \begin{cases}
.001x + y &= 1 \\
y &= 1.00
\end{cases}
\end{align*}
\]

The solution for the last system is \(x = 0, y = 1\) which is wildly inaccurate (and the condition number is \(\approx 2002\)).

This problem can be avoided using \textit{partial pivoting}. Instead of pivoting on the first non-zero element, \textit{pivot on the largest pivot (in absolute value) among those available in the column}. In the example above, pivot on the \(x\), which will require a permutate first:

\[
\begin{align*}
\begin{cases}
x + y &= 2 \\
.001x + y &= 1
\end{cases}
\quad \begin{cases}
x + y &= 2 \\
.999y &= .998
\end{cases}
\quad \begin{cases}
x + y &= 2 \\
y &= 1.00
\end{cases}
\end{align*}
\]

where the third system is the one obtained after rounding. The solution is a fairly accurate \(x = 1.00, y = 1.00\) (and the condition number is 4).